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Nonlinear baroclinic instability of a continuous zonal flow of viscous fluid

By P. G. DRAZIN
School of Mathematics, University of Bristol

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Nonlinear instability of a zonal flow of slightly viscous Boussinesq fluid in a rapidly rotating frame is studied mathematically by the method of normal mode cascade, the flow being along a rectangular channel with horizontal and vertical rigid walls. Viscosity is represented approximately by supposing that its only effects occur in Ekman layers near the top and bottom walls of the channel, after the linear model of Barcilon. Self-interaction of one slightly unstable mode is found to lead to equilibration with supercritical instability. Also, interactions of two slightly unstable modes plausibly lead to equilibration. These results are related to the literature of experiments on differentially heated, rotating annuli.

1. Introduction

Recently the author (Drazin 1970) took Eady's model of baroclinic instability of a shear flow of inviscid non-conducting Boussinesq fluid in a rapidly rotating frame and extended it by including nonlinearity. Applications of the results to instability of the westerly winds and to experiments on a differentially heated, rotating annulus of liquid were limited by the absence of viscosity or another dissipative mechanism in the model. However, there is no entirely satisfactory treatment of linear baroclinic instability of viscous fluid for which the disturbance and the flow itself are exact solutions of the equations of motion and the boundary conditions. "In order to reduce the mathematical complexity" Barcilon (1964, p. 293) made "several simplifications" to get a tractable model of linear baroclinic instability of viscous fluid, but they seemed to prevent "direct comparison between...results and the experiments". Assuming that the kinematic viscosity and thermal diffusivity were small and of the same order of magnitude, he included Ekman layers on the rigid top and bottom of the square channel of his model, but neglected the buoyancy layers on the vertical side walls in order to simplify the mathematics. In this way he let thermal diffusivity drop entirely out of his model. In spite of these assumptions and of Barcilon's disclaimer, after a simple generalization of his results for basic flows along channels of rectangular instead of square section, his theory agrees quite well with experiments on a differentially heated, rotating annulus with a free surface and is, indeed, comparable to other linear theories (Fowlis & Hide 1965; Kaiser 1970, figure 4). Barcilon's linear theory is also relatively simple, so it seems suitable to begin a study of nonlinear baroclinic instability of real fluid.

Barcilon (1964) used boundary-layer theory to replace the thin Ekman layers by modified boundaıy conditions on the steady geostrophic solution for inviscid fluid. These conditions are equally valid for nonlinear unsteady instability in the geostrophic limit, because acceleration and inertia play no role in an Ekman layer. Thus, following Barcilon, we may add the viscous terms to the nonlinear problem for inviscid fluid. This gives the problem treated by Drazin (1970, equations (27), (9), (11) and (12)) together with extra linear viscous terms in the top and bottom boundary conditions, as follows:

$$\beta_\delta z K p_x = -\left\{\frac{\partial}{\partial t}+(\beta-\beta_\delta)z\frac{\partial}{\partial x}\right\}Kp-(\beta-\beta_\delta)\left(\frac{\partial}{\partial t}+\beta z\frac{\partial}{\partial x}\right)(p_{xx}+p_{yy})$$
$$-\beta\mathbf{u}.\nabla\{p_{zz}+\beta(p_{xx}+p_{yy})\}, \tag{1}$$

$$u=-p_y,\quad v=p_x,\quad w=\epsilon\left\{p_x-zp_{zx}-\beta^{-1}p_{zt}-\frac{\partial(p,p_z)}{\partial(x,y)}\right\},\quad \theta=p_z; \tag{2}$$

$$p_x=0\quad\text{at}\quad y=\pm\tfrac{1}{2}h, \tag{3a}$$

but the zonal average of p with respect to x satisfies

$$p_{yt}=0\quad\text{at}\quad y=\pm\tfrac{1}{2}, \tag{3b}$$

$$Lp\equiv p_x-zp_{zx}+2(\tfrac{1}{2}\delta)^{\frac{1}{2}}(h\epsilon)^{-1}z(p_{xx}+p_{yy}) \tag{4}$$

$$=\beta^{-1}p_{tz}+\frac{\partial(p,p_z)}{\partial(x,y)}\quad\text{at}\quad z=\pm\tfrac{1}{2}h. \tag{5}$$

Here the operator
$$K\equiv\frac{\partial^2}{\partial z^2}+\beta_\delta\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right), \tag{6}$$

the Ekman number
$$\delta\equiv\nu/2\Omega(b-a)^2>0, \tag{7}$$

ν is the kinematic viscosity of the fluid and β_δ is an arbitrary constant at this stage (because terms involving β_δ cancel identically in equation (1)) but will be chosen to be the value of β for given δ, ϵ, h and wavenumbers of the disturbance at which there is marginal stability; otherwise the notation of Drazin (1970) is used, so

$$\beta\equiv\alpha g(\Delta T)/4\Omega^2(b-a),\ \text{etc.} \tag{8}$$

It has been judged best to state the problem (1), (2), (3) and (5) without more ado, rather than to give its background in a long discussion, repeating points covered in the three papers to which we have already referred and in many other papers. However, it may be helpful to say here that the problem is one of instability of the basic flow with velocity $\mathbf{U}=z\mathbf{i}$ and temperature $\Theta=z-\epsilon y$ in the channel with section $-\frac{1}{2}\leqslant y\leqslant\frac{1}{2}$, $-\frac{1}{2}h\leqslant z\leqslant\frac{1}{2}h$ after choice of dimensionless quantities based on the length scale $(b-a)$ and time scale $(2\Omega\epsilon)^{-1}$. In the geostrophic limit as $\epsilon\to 0$, the equations for the perturbations u, v, w, p and θ of velocity, pressure and temperature give equations (1) and (2). The boundary conditions of no normal flux on the rigid walls then give equations (3) and (5) on taking Ekman layers at $z=\pm\frac{1}{2}h$. Instability can be seen thereby to be governed by the dimensionless numbers β, δ and ϵ, which is the slope of the isotherms of the basic flow. The approximations of the model are that $\epsilon^2 \ll (\frac{1}{2}\delta)^{\frac{1}{2}} \ll 1$ in order that the flow should be quasi-geostrophic and that the thin Ekman layers should be more significant than non-geostrophic effects.

We may now proceed to use the method of normal mode cascade to find the nonlinear self-interaction of one linear mode and the interactions of two linear modes. Again, the ideas of this method have been established over the last thirty years by Landau, Meksyn, Stuart, Palm, Watson and others (cf. Segel 1966) in many problems of hydrodynamic stability, and the details involve a lot of algebra, calculus and trigonometry, so we shall suppress much of the work to emphasize the results. However, first we shall have to review Barcilon's linear theory to build a foundation for the nonlinear theory.

2. Linear solution

After linearization of equation (1) and the boundary conditions (3), the solution can be resolved into independent normal modes of the form

$$p = \epsilon^{ik(x-ct)} \sin lY(Q\cosh 2qz + R\sinh 2qz), \tag{9}$$

where k is a real wavenumber, l/π is the positive integral number of anti-nodes between the vertical walls, $Y \equiv y+\frac{1}{2}$, $q \equiv \frac{1}{2}\{\beta(k^2+l^2)\}^{\frac{1}{2}}$, Q and R are arbitrary constants and $c = c_r + ic_i$ is an eigenvalue determining the stability of the mode. This interior solution is the same as that for the inviscid problem. However, the viscous boundary conditions (5) give

$$\frac{R}{Q} = \frac{i\lambda + 2ch^{-1}s\beta^{-1}\tanh s}{s-\tanh s} \tag{10}$$

and

$$\begin{aligned} c = h\beta(4s\tanh s)^{-1}\{&-i\lambda(1+\tanh^2 s) \\ &\pm[-\lambda^2(1+\tanh^2 s)^2 + 4\lambda^2\tanh^2 s + 4\tanh s(s-\tanh s)(s\tanh s - 1)]^{\frac{1}{2}}\}, \end{aligned} \tag{11}$$

where $s \equiv hq$ and $\lambda \equiv (\frac{1}{2}\delta)^{\frac{1}{2}}(k^2+l^2)/k\epsilon$. It can be seen that marginal stability, arising as $kc_i \to 0^+$, occurs when $c = 0$, i.e. the principle of exchange of stabilities is valid. The mode is marginally stable if and only if $s = s_\delta$, where s_δ is defined as the positive root of

$$(s_\delta - \tanh s_\delta)(s_\delta \tanh s_\delta - 1) + \lambda^2 \tanh s_\delta = 0. \tag{12}$$

This gives marginal stability for fixed k, l, δ and ϵ when

$$\beta = \beta_\delta(k,l) \equiv 4s_\delta^2/h^2(k^2+l^2). \tag{13}$$

As $\delta \to 0$ along each curve $\beta = \beta_\delta$ in the δ^{-1}, β plane for fixed h, k and l, there is an upper branch on which $\beta_\delta \to 4s_0^2/h^2(k^2+l^2) \equiv \beta_0$, where $s_0 \doteqdot 1{\cdot}2$ is the positive root of $s_0 \tanh s_0 = 1$, and a lower branch on which $6\delta(k^2+l^2) \sim h^2k^2\beta_\delta\epsilon^2$. At marginal stability, we may write

$$p = p' \equiv A(t)\sin lY\left(\cos kx \cosh 2q_\delta z - \frac{\lambda}{s_\delta - \tanh s_\delta}\sin kx \sinh 2q_\delta z\right). \tag{14}$$

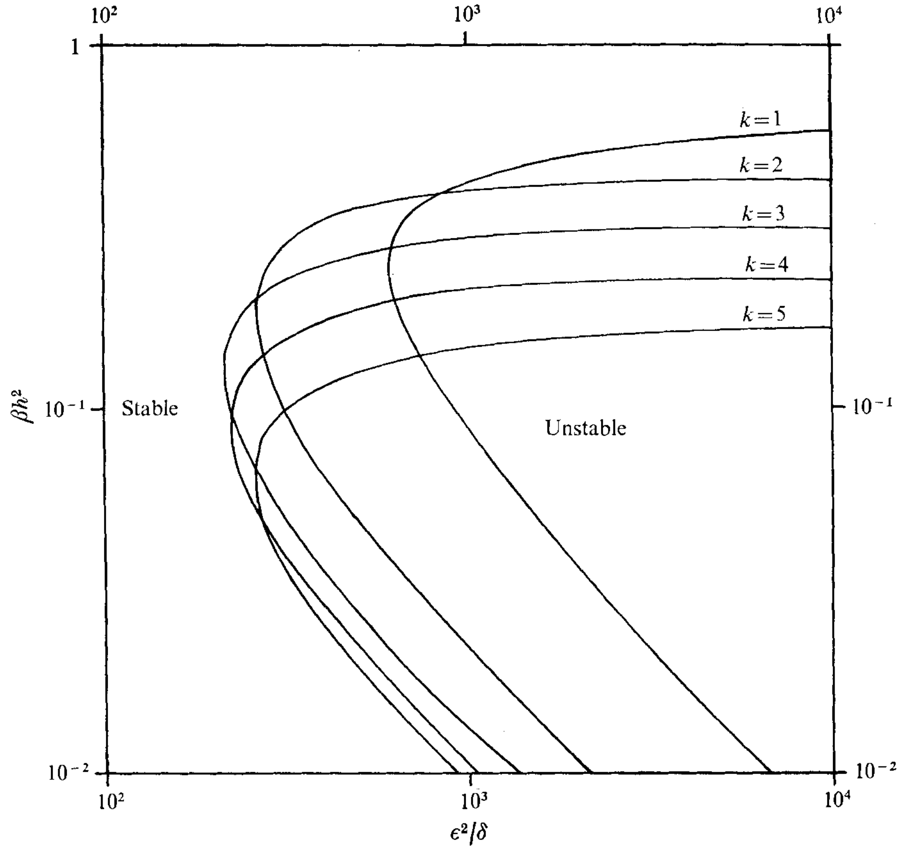

FIGURE 1. Curves of marginal stability for modes with $k = 1, 2, 3, 4, 5$ and $l = \pi$.

Barcilon applied the curves (12) of marginal stability for the modes to the instability of a differentially heated, rotating annulus by making the narrow-gap approximation that the distance $b-a$ between the outer and inner walls of the annulus is much less than the inner radius a. This application of the unbounded rectilinear configuration of the stability problem to the bounded cylindrical configuration of the annulus renders the wavenumber discrete, such that

$$(b+a)\,k/2(b-a) = n = 1, 2, 3, \ldots \tag{15}$$

is the number of waves around the annulus. (The value $n = 0$ is ignored because when $k = 0$ there is no growth of instability.) Then the curves (12) in the δ^{-1}, β plane have an envelope, along which $l = \pi$ and n varies, marking the transition between values of β and δ at which all modes are stable and values at which at least one mode is unstable, as shown in figure 1. We have followed Barcilon (1964, figure 4) in depicting the curves only for $k = n$, i.e. $b/a = 3$, but they are similar to those for other ratios of the radii. Kaiser (1970) carefully discusses the relation between the theoretical curve of marginal stability and experimental results, though it is not clear what aspect ratio h he assumes for the theoretical curves.

3. Self-interaction of a single unstable mode

In general one and only one mode is unstable in a flow corresponding to a point of the δ^{-1}, β plane just inside the envelope of marginal stability. So we assume that in such a flow the unstable mode will grow exponentially at first but later in accord with its nonlinear self-interaction. All the other modes, being exponentially damped, are plausibly negligible for all time. This is the basis for the method of normal mode cascade, whereby we shall iterate the solution (14) as $\beta \to \beta_\delta$ for fixed δ, ϵ, h, k and l. The iteration is similar to that of Drazin (1970) for the inviscid form of the present problem. The analysis is simplified if one recognizes at the outset that here a slowly growing nonlinear disturbance will be such that $\dot{A} \equiv dA/dt$, $(\beta-\beta_\delta)A$ and A^3 are of the same order of magnitude. That $\dot{A}$ and $(\beta-\beta_\delta)A$ are of the same order follows from the linear theory, because (11) gives $kc = O(\beta-\beta_\delta)$ as $\beta \to \beta_\delta$. The nonlinear approximation of the third degree in A is necessary to show that $\dot{A}$ and A^3 are of the same order of magnitude, but we shall assume this result for the present and confirm it *ex post facto* in equation (33).

Thus we put
$$p = p' + p'' + p''' + \dots, \tag{16}$$
where p'' is of order A^2, p''' of order A^3 etc., in order to iterate the fundamental linear solution p' for marginal stability in the full problem (1), (3) and (5). This gives
$$Kp''_x = 0, \tag{17}$$
$$p''_{yt} = 0 \quad \text{at} \quad y = \pm\tfrac{1}{2}, \tag{18}$$
$$Lp'' = h^{-1}kls_\delta(s_\delta - \tanh s_\delta)^{-1}\lambda A^2 \sin 2lY \quad \text{at} \quad z = \pm\tfrac{1}{2}h, \tag{19}$$
because in fact the right-hand side of condition (19) is independent of x and so condition (3*b*) applies here. It follows that
$$p'' = p''(y,z,t), \tag{20}$$
$$u''_t = 0 \quad \text{at} \quad y = \pm\tfrac{1}{2}, \tag{21}$$
$$u''_y = \mp \frac{4ls_\delta^3 A^2 \sin 2lY}{h^3\beta_\delta(s_\delta - \tanh s_\delta)} \quad \text{at} \quad z = \pm\tfrac{1}{2}h. \tag{22}$$
Here $u'' = -p''_y$ will have to be specified fully at the next iteration. The product of p' and an arbitrary constant could be added to p'' without loss of generality, by redefinition of the fundamental amplitude A if necessary.

The next iteration, after some manipulation, gives
$$Kp'''_x = 4h^{-2}s_\delta^2\beta_\delta^{-1}(\beta-\beta_\delta)p'_x + z^{-1}(u''_{zz} + \beta_\delta u''_{yy})p'_x, \tag{23}$$
$$p'''_x = 0 \text{ etc.} \quad \text{at} \quad y = \pm\tfrac{1}{2}, \tag{24}$$
$$Lp''' = 2h^{-1}s_\delta\beta_\delta^{-1}\cosh s_\delta \dot{A}\left(\pm\tanh s_\delta \cos kx - \frac{\lambda}{s_\delta - \tanh s_\delta}\sin kx\right) + p'_{zx}u'' - p'_x u''_z \quad \text{at} \quad z = \pm\tfrac{1}{2}h. \tag{25}$$

It can now be seen that
$$u''_{zz} + \beta_\delta u''_{yy} = 0 \tag{26}$$

in order that p'' be bounded, because otherwise the right-hand side of (23) would lead to a particular integral of p''' growing in x like xp'. Thus (21), (22) and (26) determine u''. Seeking a solution that is continuous on the boundary and that vanishes with $A(t)$, we find that

$$u'' = -\frac{2s_\delta^3 A^2}{h^3\beta_\delta(s_\delta - \tanh s_\delta)}\sum_{j=0}^{\infty} a_{2j+1}\sin(2j+1)\pi Y \frac{\sinh(2j+1)\pi\beta_\delta^{\frac{1}{2}} z}{\sinh(j+\frac{1}{2})\pi h\beta_\delta^{\frac{1}{2}}}, \tag{27}$$

after some Fourier analysis, where the coefficients of the sine series for $1-\cos 2lY$ are

$$a_{2j+1} = -2l^2/(j+\tfrac{1}{2})\pi\{(j+\tfrac{1}{2})^2\pi^2 - l^2\} \quad \text{for} \quad j = 0, 1, 2, \ldots. \tag{28}$$

The inhomogeneous system (23)–(25) is now specified fully. It has a solution p''' only if the inhomogeneous terms on the right-hand sides satisfy a solubility condition. This condition can be found by use of a solution of the adjoint homogeneous system,

$$p^* \equiv \left(\cos kx\cosh 2q_\delta z + \frac{\lambda}{s_\delta - \tanh s_\delta}\sin kx\sinh 2q_\delta z\right)\sin lY. \tag{29}$$

For one can then easily show that

$$0 = \frac{k}{2\pi h}\int_{-\frac{1}{2}h}^{\frac{1}{2}h}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_0^{2\pi/k} p'''Kp_x^*\,dx\,dy\,dz \tag{30}$$

$$= \frac{k}{2\pi h}\left\{\int_{-\frac{1}{2}h}^{\frac{1}{2}h}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_0^{2\pi/k} p_x^*\,Kp_x'''\,dx\,dy\,dz + \int_{-\frac{1}{2}h}^{\frac{1}{2}h}\int_0^{2\pi/k}[\beta_\delta p_{xy}^* p_x''']_{y=-\frac{1}{2}}^{\frac{1}{2}}\,dx\,dz + \int_{-\frac{1}{2}h}^{\frac{1}{2}}\int_0^{2\pi/k}[z^{-1}p_x^*\,Lp''']_{z=-\frac{1}{2}h}^{\frac{1}{2}h}\,dx\,dy\right\}, \tag{31}$$

on integration by parts and on use of the periodicity in x; this becomes

$$\begin{aligned} 0 = {} & h^{-3}k^2s_\delta^2\beta_\delta^{-1}(\beta-\beta_\delta)A\left\{\int_{-\frac{1}{2}h}^{\frac{1}{2}h}\cosh^2 2q_\delta z - \frac{\lambda^2\sinh^2 2q_\delta z}{(s_\delta-\tanh s_\delta)}\,dz\right\} \\ & + \frac{2ks_\delta(\cosh^2 s_\delta + \sinh^2 s_\delta)\lambda}{h^3\beta_\delta(s_\delta-\tanh s_\delta)}\dot{A} + \frac{2k^2s_\delta^3\cosh^2 s_\delta A^3}{h^6\beta_\delta(s_\delta-\tanh s_\delta)^2} \\ & \times\left\{3s_\delta(1+\tanh^2 s_\delta - 2s_\delta\tanh s_\delta) + (s_\delta + s_\delta\tanh^2 s_\delta - 2\tanh s_\delta)\right. \\ & \left.\times\sum_{j=0}^{\infty} a_{2j+1}^2(j+\tfrac{1}{2})\pi h\beta_\delta^{\frac{1}{2}}\coth[(j+\tfrac{1}{2})\pi h\beta_\delta^{\frac{1}{2}}]\right\}. \end{aligned} \tag{32}$$

Now the Landau equation, $\dot{A} = C_1A - C_2A^3$, (33)

follows on neglect of terms of order A^4; here

$$C_1 \equiv \frac{hk(\beta_\delta-\beta)}{4\lambda(1+\tanh^2 s_\delta)}\{(s_\delta-\tanh s_\delta)(s_\delta - s_\delta\tanh^2 s_\delta + \tanh s_\delta) - \lambda^2(\tanh s_\delta - s_\delta + s_\delta\tanh^2 s_\delta)\}, \tag{34}$$

$$C_2 \equiv \frac{ks_\delta^2}{h^3\lambda(1+\tanh^2 s_\delta)(s_\delta-\tanh s_\delta)}\left\{6\lambda^2\tanh s_\delta + (s_\delta + s_\delta\tanh^2 s_\delta - 2\tanh s_\delta)\times\sum_{j=0}^{\infty}a_{2j+1}^2[(j+\tfrac{1}{2})\pi h\beta_\delta^{\frac{1}{2}}\coth\{(j+\tfrac{1}{2})\pi h\beta_\delta^{\frac{1}{2}}\} - 1]\right\}. \tag{35}$$

The linear theory agrees with the Landau equation (33) because (11) gives $kc_i \sim C_1$ as $\beta \to \beta_\delta$. The Landau constant

$$C_2 \sim \frac{ks_0^4}{h^3\lambda(1+s_0^2)} \sum_{j=0}^{\infty} a_{2j+1}[(j+\tfrac{1}{2})\pi h\beta_0^{\frac{1}{2}} \coth\{(j+\tfrac{1}{2})\pi h\beta_0^{\frac{1}{2}}\} - 1] > 0, \tag{36}$$

as $\delta \to 0$, $\beta \to \beta_0$ on the upper branch of the curve of marginal stability for the mode with given wavenumbers k, l. Also

$$C_2 \sim 6(\tfrac{1}{2}\delta)^{\frac{3}{2}}(3k^2+l^2)/h^3\epsilon > 0, \tag{37}$$

as $\delta \to 0$, $\beta \to 0$ on the lower branch of the 'marginal curve'. Numerical calculations show that $C_2 > 0$ all along the marginal curves. It follows now from the elementary solution of the Landau equation that there is always supercritical instability such that any slightly unstable solution of equation (33) equilibrates, so that

$$A \to A_e \equiv (C_1/C_2)^{\frac{1}{2}} \quad \text{as} \quad t \to +\infty, \tag{38}$$

whatever the initial small amplitude of the disturbance may be. Note that $A_e \propto |\beta - \beta_\delta|^{\frac{1}{2}}$ is small.

Equation (2) implies that $u'' = -p''_y$ depends only upon y, z and t and that $v'' = p''_x = 0$. Thus the second harmonic is not excited by second-degree interactions, which happen only to distort the mean zonal flow in the viscous problem as well as the inviscid one (Drazin 1970).

4. Interaction of two modes

We have described how on the marginal envelope for a rotating annulus $l = \pi$ and k increases discontinuously from the upper branch where $n = 1$ down to the lower branch where $n \to \infty$. Thus there are exceptional points on the envelope near which two linear modes are unstable, modes whose numbers n of waves differ by unity. Near such an exceptional point the two slightly unstable modes and their nonlinear interactions may be significant, but it is plausible that all other modes are negligible. So we take here a primary disturbance with components of different wavenumbers k_1 and k_2 but the same values of h, l, β and δ/ϵ^2, namely

$$p' \equiv A_1 p_1 + A_2 p_2, \tag{39}$$

where
$$p_r \equiv \left(\cos k_r x \cosh 2q_r z - \frac{\lambda_r}{s_r - \tanh s_r} \sin k_r x \sinh 2q_r z\right) \sin lY \tag{40}$$

and $A_r = A_r(t)$ for $r = 1, 2$. The subscripts are used to specify that a quantity is related to one of the distinct wavenumbers k_1 and k_2. Following the methods of § 3 on self-interaction, one may show that the fundamental (39) gives

$$p'' = A_1^2 p_{11} + A_1 A_2 p_{12} + A_2^2 p_{22}, \tag{41}$$

where p_{11} and p_{22} can be found from (27), and

$$p_{12} = h^{-1}\{\operatorname{sech} s_+[a_+ \cos k_+ x \sinh(2h^{-1}s_+ z) + b_+ \sin k_+ x \cosh(2h^{-1}s_+ z)] + \operatorname{sech} s_-[a_- \cos k_- x \sinh(2h^{-1}s_- z) + b_- \sin k_- x \cosh(2h^{-1}s_- z)]\} \sin 2lY. \tag{42}$$

k_1	k_2	$h^2\beta_\delta$	ϵ^2/δ	h^3C_{31}	h^3C_{32}
1	2	0·373	810	1·39	23·4
2	3	0·204	252	3·28	18·2
3	4	0·103	218	10·4	19·2
4	5	0·051	277	19·0	21·0
5	6	0·028	400	24·5	25·2

TABLE 1

Here
$$k_\pm \equiv k_1 \pm k_2, \quad s_\pm \equiv \tfrac{1}{2}\{h^2\beta_\delta(k_\pm^2+4l^2)\}^{\frac{1}{2}} \tag{43}$$

and the constants $a_\pm$ and $b_\pm$ are found explicitly by substitution into the boundary conditions at $z = \pm\frac{1}{2}h$. It can thence be shown at length that

$$\left.\begin{aligned} \dot{A}_1 &= C_{11}A_1 - C_{21}A_1^3 - C_{31}A_1A_2^2, \\ \dot{A}_2 &= C_{12}A_2 - C_{22}A_2^3 - C_{32}A_1^2A_2, \end{aligned}\right\} \tag{44}$$

where C_{11}, C_{21}, C_{12} and C_{32} can be found from (34) and (35), and the method of § 3 gives

$$C_{31} = \frac{s_2^3(s_1-\tanh s_1)}{s_1^3(s_2-\tanh s_2)}C_{21} + D_{31}/h \tag{45}$$

for a certain constant D_{31} which can be found in explicit but intricate terms of $a_\pm$, $b_\pm$ etc. One can find C_{32} from an expression analogous to (45).

Fortunately the qualitative behaviour of systems of the form (44) when the C's are positive has already been discussed by Segel & Stuart (1962). The solutions $A_1(t)$ and $A_2(t)$ are such that either $A_1 \to A_{e1}$, $A_2 \to 0$ or $A_1 \to 0$, $A_2 \to A_{e2}$ as $t \to +\infty$ according to the initial conditions, where $A_{er} \equiv (C_{1r}/C_{2r})^{\frac{1}{2}}$ for $r = 1, 2$. Thus the primary disturbance equilibrates towards one of its components after an initial period in which both components grow exponentially.

The algebraic intricacy of (45) makes it difficult to find the sign of C_{31} analytically. The expression involves essentially five independent parameters, namely k_1, k_2, l, $h^2\beta_\delta$ and ϵ^2/δ. However, $l = \pi$ and there is a given value of ϵ^2/δ for each $h^2\beta_\delta$ on the marginal envelope; moreover, for an annulus, we are interested in those points of the marginal envelope common to wavenumbers k_1 and k_2 whose ratio is $(n+1)/n$. So we took $k_1 = n+1$, $k_2 = n$ and used a computer to evaluate C_{31} and C_{32} at the points of the marginal envelope where the marginal curves of k_1 and k_2 intersect for $n = 1, 2, 3, 4, 5$ and $l = \pi$. The results are given in table 1. It can be seen that C_{31} and C_{32} are positive. Various computations of the coefficients elsewhere in the ϵ^2/δ, β plane gave positive values near the marginal envelope. However, C_{31} may become infinite and then negative where the wavenumbers k_+, $2l$ or k_-, $2l$ correspond to a marginally stable linear mode, i.e. where there is a second-order resonant interaction. This only happens on the unstable side of the marginal envelope, because that envelope corresponds to a single value of the wavenumber in the y direction, namely $l = \pi$. Now weak nonlinear instability only arises near the stability boundary, and therefore our theory is not strictly valid where the resonance occurs.

There is also a third-order resonant interaction if $|k_1 \pm 2k_2| = |k_1|$ or $|k_2|$, in which case the calculation of C_{31} or C_{32} must be modified somewhat. We may require that $k_1 > k_2 > 0$ without loss of generality, so that such a resonance may occur only if $k_1/k_2 = 3$. However, in applications to an annulus $k_1/k_2 = (n+1)/n$, so this resonance never arises and may be ignored.

We have now justified direct use of the study of Segel & Stuart (1962), and conclude that equilibration at one or other of the component modes of the fundamental occurs, such that the initial preponderance of one component leads to the eventual dominance of that component.

5. Conclusions

We have extended our earlier work (Drazin 1970) on nonlinear baroclinic instability of inviscid fluid by approximation of viscous effects. This led to the amplitude equation (33) instead of a time-reversible equation of the form $\ddot{A} = C_1 A - C_2 A^3$. For an inviscid fluid it is found that

$$C_2 = \tfrac{1}{4}k^2 l^2 \beta_0^2 \left\{1 + \sum_{j=0}^{\infty} \frac{b_{2j+1}^2 \tanh (j+\frac{1}{2})\pi h \beta_0^{\frac{1}{2}}}{(j+\frac{1}{2})\pi h \beta_0^{\frac{1}{2}}}\right\},$$

where

$$b_{2j+1} = -2(j+\tfrac{1}{2})\pi/\{(j+\tfrac{1}{2})^2\pi^2 - l^2\}.$$

(All the Fourier series of Drazin (1970) are wrong owing to unjustifiable differentiation term by term.) Thus the form of the Landau equation and the Landau constant of the inviscid limit of the present model are very different from those of the inviscid model. This difference is due to a non-uniform limit, the order of the limits as $\beta \to \beta_\delta$ and $\delta \to 0$ being significant. Indeed, the linear equation (11) gives $c_i \propto \delta^{-\frac{1}{2}}$ as $\beta \to \beta_\delta$ for fixed $\delta \neq 0$ but inviscid theory with $\delta = 0$ gives c_i well-behaved for all β.

Pedlosky (1970, 1971) has recently analysed the nature of this singular perturbation for a two-layer basic flow, finding a third-order differential system for the amplitude that reduces to the first-order viscous equation or the second-order inviscid equation in the appropriate limits. Such a discontinuous velocity profile in a slightly viscous fluid strictly needs a correction owing to the Ekman layer at the interface between the two layers, but this does not modify the qualitative stability characteristics (Pedlosky 1970, p. 18). Indeed, the continuous profile $U = z$ treated here exhibits instability qualitatively similar to that of the two-layer model.

However, we have emphasized two aspects of particular value in modelling experiments on a differentially heated, rotating annulus, the marginal envelope and the interaction of two components. With the basic linear flow, which is itself the thermal wind associated with the basic temperature field, Barcilon (1964) referred stability characteristics to the ϵ^2/δ, β plane, finding its knee-shaped marginal envelope. His representation of viscous effects only by matching Ekman suction with inviscid interior flow near the top and bottom walls is justifiable in the limit as $\epsilon \to 0$ for fixed δ/ϵ^2. The vertical velocity in the interior is $w = O(\epsilon)$ as $\epsilon \to 0$ in the geostrophic limit and is matched with the Ekman suction of velocity $w = O(\delta^{\frac{1}{2}})$ as $\delta \to 0$; this gives a correction of order $\delta^{\frac{1}{2}}/\epsilon$ to the stability problem. In the interior $u, v, \theta = O(1)$ as $\epsilon \to 0$, so matching of these variables gives a correction of order $\delta^{\frac{1}{2}}$, which is negligible. Ageostrophic effects in the interior are of order ϵ, and are neglected. It happens that the vertical velocity of the interior solution vanishes on the side walls in Barcilon's linear solution; so matching of the no-slip conditions gives a correction only of order $\delta^{\frac{1}{2}}$, which is negligible. In our nonlinear problem it can be seen also that

$$w'' = \epsilon\left(p''_x - zp''_{zx} - \frac{\partial(p', p'_z)}{\partial(x, y)}\right),$$

$$w''' = \epsilon\left(p'''_x - zp'''_{zx} - \beta_\delta^{-1} p'_{zt} - \frac{\partial(p', p''_z)}{\partial(x, y)} - \frac{\partial(p'', p'_z)}{\partial(x, y)}\right)$$

vanish on the side walls $y = \pm\frac{1}{2}$, so that the Landau equation (33) includes the most significant viscous terms for the given basic flow. The temperature perturbation $\theta' = p'_z$ vanishes at the side walls, $y = \pm\frac{1}{2}$, so that the linear solution corresponds to side walls maintained at steady temperatures. However, it can be seen from (27) that $\theta'' = p''_z$ does not vanish for all z at $y = \pm\frac{1}{2}$, so a thermal boundary layer would arise. This is a weakness of the model for real fluids of finite Prandtl number in a channel of bounded cross-section. Solution of the appropriate boundary-value problem would lead, with difficulty, to a corrected value of the Landau constant C_2. One might speculate that this correction would increase C_2, because the thermal diffusivity would damp the temperature field of the disturbance rather than unleash any potential energy from the basic flow as the amplitude grew, much as viscosity serves to dissipate the disturbance as the amplitude grows so that equilibration ensues.

In § 4 we considered the exceptional case where two modes were both slightly unstable. Now it is observed that the change from one wavenumber n to the next in the rotating annulus as the angular velocity Ω increases occurs at different points in the $\epsilon^2/\delta, \beta$ plane according to whether Ω is decreasing or increasing. There is 'hysteresis' whereby a mode is preferred if it has been present earlier in time. This is consistent with the interaction of two modes described in § 4, the initial values of the amplitudes of two competing components influencing which component the disturbance ultimately equilibrates to.

Because of the differences between the theoretical and experimental models (cf. Hide 1970, p. 203), some caution is needed in applying theory to experiments, and more is necessary for application to the atmosphere. None the less, Eady's model, as developed by Barcilon for viscous fluid and here for nonlinear disturbances, does provide encouraging comparison with experiments on a differentially heated, rotating annulus.

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